RE-INTRODUCTION TO MIXED MODELS

DEFINING A MIXED MODEL

The working definition of a mixed model is a model with some fixed effects parameters and more than one variance and/or covariance component.

Example 1: The one-way analysis of variance model, with unequal variances per treatment group, is a good example of a mixed model. In this case, both the means and variances should be modeled.

The MODEL is $Y_{ij} = \mu + \tau_i + e_{ij}$

where e_{ij} ~ independent, normal $(0, \sigma_i^2)$

$i=1,2, \ldots, t$ and $j=1,2, \ldots, n_i$.

This is a mixed model since it involves

t variance components (σ_i^2) and

t means ($\mu + \tau_i$).

Example 2: The split unit (split plot) design is another example of a mixed model. The fixed effects are a factorial treatment structure and the model contains two variance components - one for the whole units and one for sub units.

The MODEL is

$$Y_{ijk} = \mu + \tau_i + b_k + w_{ik} + \alpha_j + (\tau\alpha)_{ij} + e_{ijk}$$

where: b_k is assumed to be iid $N(0, \sigma_b^2)$,
w_{ik} is assumed to be iid $N(0, \sigma_w^2)$,
e_{ijk} is assumed to be iid $N(0, \sigma_e^2)$,
$i=1,2, \ldots, t$; $j=1,2, \ldots, a$; and $k=1,2, \ldots, b$,
and τ_i, α_j, $(\tau\alpha)_{ij}$ are fixed effects.

Thus, this mixed model involves fitting three variance components (σ^2) and t*a means ($\mu + \tau_i + \alpha_j + (\tau\alpha)_{ij}$).

Example 3: A repeated measures design would be a third example. Repeated measures designs have the structure similar to that of a split-plot design.

The treatment structure is at least a two-way factorial arrangement, where one factor is treatment and the other is time. Treatments are assigned to experiment units (whole units) and on each whole unit data are recorded during each time period (sub units). Thus, we have two sizes of units making it necessary to model two random variance components.

Because time cannot be randomly assigned we must also model the covariance(s) among time periods, which are additional random components.

FIXED vs MIXED MODELS

The Fixed Model Simply Stated

Y = fixed treatment and/or covariate effect(s)
+ random residual

The Mixed Model Simply Stated

Y = fixed treatment and/or covariate effect(s)
+ random effect(s)
+ random residual

IS AN EFFECT RANDOM OR FIXED?

A fixed effect is one that is repeatable. That is, if other scientists repeat your experiment, they would be estimating the same differences among treatment means, the same covariate regression coefficients and the same differences among regression coefficients.

A random effect is one that would not be repeatable. That is, another researcher would not (probably could not) estimate the same effects, but could estimate the variance of the effects from another sample.

CRD, 2-treatments each with 3-replicates.

$Y_{ij} = \mu + \tau_i + e_{ij}$

e_{ij} is assumed to be iid $N(0, \sigma_e^2)$

Trt		----- 1 -----			----- 2 -----		
	Rep	1	2	3	1	2	3
1	1	σ_e^2	0	0	0	0	0
1	2	0	σ_e^2	0	0	0	0
1	**3**	**0**	**0**	σ_e^2	0	0	0
2	**1**	**0**	**0**	**0**	σ_e^2	0	0
2	**2**	**0**	**0**	**0**	**0**	σ_e^2	0
2	**3**	**0**	**0**	**0**	**0**	**0**	σ_e^2

CRD, 2-treatments each with 3-replicates.

$Y_{ij} = \mu + \tau_i + e_{ij}$

e_{ij} is assumed to be iid $N(0, \sigma_i^2)$

Trt	Rep	----- 1 -----			----- 2 -----		
		1	2	3	1	2	3
1	1	σ_1^2	0	0	0	0	0
1	2	0	σ_1^2	0	0	0	0
1	3	0	0	σ_1^2	0	0	0
2	1	0	0	0	σ_2^2	0	0
2	2	0	0	0	0	σ_2^2	0
2	3	0	0	0	0	0	σ_2^2

RCBD, 3-treatments each with 3-blocks.

$Y_{ij} = \mu + \tau_i + e_{ij}$

Rep	Trt	--- 1 ---			--- 2 ---			--- 3 ---		
		1	2	3	1	2	3	1	2	3
1	1	σ_e^2	σ_{11}	σ_{11}	0	0	0	0	0	0
1	2	σ_{11}	σ_e^2	σ_{11}	0	0	0	0	0	0
1	2	σ_{11}	σ_{11}	σ_e^2	0	0	0	0	0	0
2	1	0	0	0	σ_e^2	σ_{11}	σ_{11}	0	0	0
2	2	0	0	0	σ_{11}	σ_e^2	σ_{11}	0	0	0
2	2	0	0	0	σ_{11}	σ_{11}	σ_e^2	0	0	0
2	1	0	0	0	0	0	0	σ_e^2	σ_{11}	σ_{11}
2	**2**	**0**	**0**	0	0	0	0	σ_{11}	σ_e^2	σ_{11}
2	**2**	**0**	**0**	0	0	0	0	σ_{11}	σ_{11}	σ_e^2

CRD, 2-treatments, 3-replicates each with 2 samples.

$$Y_{ijk} = \mu + \tau_i + e_{ijk}$$

Trt	R	S	1						2					
			1		2		3		1		2		3	
			1	2	1	2	1	2	1	2	1	2	1	2
1	1	1	σ_e^2	σ_{11}	0	0	0	0	0	0	0	0	0	0
1	1	2	σ_{11}	σ_e^2	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	σ_e^2	σ_{11}	0	0	0	0	0	0	0	0
1	2	2	0	0	σ_{11}	σ_e^2	0	0	0	0	0	0	0	0
1	3	1	0	0	0	0	σ_e^2	σ_{11}	0	0	0	0	0	0
1	3	2	0	0	0	0	σ_{11}	σ_e^2	**0**	**0**	**0**	**0**	**0**	**0**
2	1	1	0	0	0	0	0	0	σ_e^2	σ_{11}	0	0	0	0
2	**1**	**2**	0	0	0	0	0	0	σ_{11}	σ_e^2	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	σ_e^2	σ_{11}	**0**	**0**
2	**2**	**2**	0	0	0	0	0	0	0	0	σ_{11}	σ_e^2	**0**	**0**
2	3	1	0	0	0	0	0	0	0	0	0	0	σ_e^2	σ_{11}
2	**3**	**2**	0	0	0	0	0	0	0	0	0	0	σ_{11}	σ_e^2

Repeated measures, 2 treatments, 4 weeks and 2 experimental units per treatment.

$$Y_{ijk} = \mu + T_i + W_{ik} + (TW)_{ij} + e_{ijk}$$

Trt			**1**								**2**							
	EU		**1**				**2**				**1**				**2**			
		Wk	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
1	**1**	**1**	σ_e^2	σ_{11}	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**1**	**2**	σ_{11}	σ_e^2	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**1**	**3**	σ_{11}	σ_{11}	σ_e^2	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**1**	**4**	σ_{11}	σ_{11}	σ_{11}	σ_e^2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**2**	**1**	**0**	**0**	**0**	**0**	σ_e^2	σ_{11}	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**2**	**2**	**0**	**0**	**0**	**0**	σ_{11}	σ_e^2	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**2**	**3**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_e^2	σ_{11}	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**2**	**4**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_{11}	σ_e^2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_e^2	σ_{11}	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**
2	**1**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_e^2	σ_{11}	σ_{11}	**0**	**0**	**0**	**0**
2	**1**	**3**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_e^2	σ_{11}	**0**	**0**	**0**	**0**
2	**1**	**4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_{11}	σ_e^2	**0**	**0**	**0**	**0**
2	**2**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_e^2	σ_{11}	σ_{11}	σ_{11}
2	**2**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_e^2	σ_{11}	σ_{11}
2	**2**	**3**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_e^2	σ_{11}
2	**2**	**4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	σ_{11}	σ_{11}	σ_{11}	σ_e^2

Repeated measures, 2 treatments, 4 weeks and 2 experimental units per treatment.

$$Y_{ijk} = \mu + T_i + W_{ik} + (TW)_{ij} + e_{ijk}$$

For treatment 1:

Trt			**1**							
	EU		**1**				**2**			
		Wk	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
1	**1**	**1**	σe^2	$\rho\sigma e^2$	$\rho^2\sigma e^2$	$\rho^3\sigma e^2$	**0**	**0**	**0**	**0**
1	**1**	**2**	$\rho\sigma e^2$	σe^2	$\rho\sigma e^2$	$\rho^2\sigma e^2$	**0**	**0**	**0**	**0**
1	**1**	**3**	$\rho^2\sigma_e^2$	$\rho\sigma e^2$	σe^2	$\rho\sigma e^2$	**0**	**0**	**0**	**0**
1	**1**	**4**	$\rho^3\sigma e^2$	$\rho^2\sigma e^2$	$\rho\sigma e^2$	σe^2	**0**	**0**	**0**	**0**
1	**2**	**1**	**0**	**0**	**0**	**0**	σe^2	$\rho\sigma e^2$	$\rho^2\sigma e^2$	$\rho^3\sigma e^2$
1	**2**	**2**	**0**	**0**	**0**	**0**	$\rho\sigma e^2$	σe^2	$\rho\sigma e^2$	$\rho^2\sigma e^2$
1	**2**	**3**	**0**	**0**	**0**	**0**	$\rho^2\sigma_e^2$	$\rho\sigma e^2$	σe^2	$\rho\sigma e^2$
1	**2**	**4**	**0**	**0**	**0**	**0**	$\rho^3\sigma e^2$	$\rho^2\sigma e^2$	$\rho\sigma e^2$	σe^2